M.Sc. - II (Mathematics) (New CBCS Pattern) Semester-III

PSCMTH11 - Complex Analysis

P. Pages : 3 **GUG/S/25/13755**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each questions carry equal marks.

UNIT – I

1. a) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some $\in$ neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that **10**

i) The first-order partial derivatives of the functions u and v with respect to x and y exist every where in the neighborhood.

ii) Those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations. $u_x = v_y, u_y = -v_x$ at (x_0, y_0)

Then prove that $f'(z_0)$ exists, its value being $f'(z_0) = u_x + iv_x$ where the right-hand side is to be evaluated at (x_0, y_0)

b) Suppose that **10**

i) A function f is analytic throughout a domain D.

ii) f(z) = 0 at each point z of a domain or line segment contained in D.

Then prove that $f \equiv 0$ in D: that is, f(z) is identically equal to zero throughout D.

OR

c) Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then prove that $\overline{f(z)} = f(\overline{z})$ for each point z in the domain if and only if f(x) is real for each point x on the segment. **10**

d) Prove that if a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D. **10**

UNIT – II

2. a) Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then prove that $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0}$ **10**

b) Prove that if a function f is entire and bonded in the complex plane, then f(z) is constant throughout the plane. **10**

OR

c) Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0. Then prove that f(z) has the power series representation **10**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ (|z - z_0| < R_0),$$

Where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, ...)$$

d) Prove that if a power series **10**

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Converges when $z = z_1 (z_1 \neq z_0)$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$

UNIT – III

3. a) Prove that if a function f is analytic every where in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then $\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$. **10**

b) Given a function f and a point z_0 suppose that **10**

i) f is analytic at z_0

ii) $f(z_0) = 0$ but f(z) is not identically equal to zero in any neighborhood of z_0.

Then prove that $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \in$ of z_0.

OR

c) Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) = 0$, then prove that z_0 is a simple pole of the quotient $\frac{p(z)}{q(z)}$ and $\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$. **10**

d) Suppose that **10**

i) A function f(z) is analytic at all points in the upper half plane $y \geq 0$ that are exterior to a circle $|z| = R_0$:

ii) C_R denotes a semicircle $z = Re^{i\theta} (0 \leq \theta \leq \pi)$, where $R > R_0$

iii) For all points z on C_R, there is a positive constant M_R such that $|f(z)| \leq M_R$ and $\lim_{R \to \infty} M_R = 0$

Then prove that for every positive constant a, $\lim_{R \to \infty} \int_{C_R} f(z) e^{iaz} dz = 0$.

UNIT – IV

4. a) Find a linear transformation that maps the strip x > 0, 0 < y < 2 onto the strip $-1 < u < 1,\ v > 0$. **10**

b) Let the circle $|z| = 1$ have a positive, or counterclockwise, orientation. Determine the orientation of its image under the transformation $w = \frac{1}{z}$. **10**

OR

c) Show that the transformation $w = \sin z$ is a one to one mapping of the semi-infinite strip $\frac{-\pi}{2} < x < \frac{\pi}{2},\ y \geq 0$ in the z plane onto the upper half $v \geq 0$ of the w plane. **10**

d) Find the image of the semi-infinite strip $x \geq 0,\ 0 \leq y \leq \pi$ under the transformation $w = \exp z$, and label corresponding portions of the boundaries. **10**

5. a) Show that the function $f(z) = \frac{1}{z^2}$ is analytic at every non zero point z of the complex plane. **5**

b) Prove that the absolute convergence of a series of complex numbers implies the convergence of that series. **5**

c) Find the integral $\int_C \frac{e^z - 1}{z^4} dz$ **5**

Where C is the positively oriented unit circle $|z| = 1$.

d) Find the fixed point of the transformation. **5**

$$w = \frac{6z - 9}{z}$$

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